Nonconservative Lagrangian and Hamiltonian mechanics

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Traditional Lagrangian and Hamiltonian mechanics cannot be used with nonconservative forces such as friction. A method is proposed that uses a Lagrangian containing derivatives of fractional order. A direct calculation gives an Euler-Lagrange equation of motion for nonconservative forces. Conjugate momenta are defined and Hamilton's equations are derived using generalized classical mechanics with fractional and higher-order derivatives. The method is applied to the case of a classical frictional force proportional to velocity.

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I. INTRODUCTION

It is a strange paradox that the most advanced methods of classical mechanics deal only with conservative systems, while almost all classical processes observed in the physical world are nonconservative. Conservative systems are time reversible by definition, while nonconservative systems exhibit the familiar arrow of time due to irreversible dissipative effects such as friction. Friction and irreversibility are also found in the quantum world, since all systems that we think of as classical are fundamentally quantum. Even at the microscopic level, there is dissipation in every nonequilibrium or fluctuating process, including dissipative tunneling [1], electromagnetic cavity radiation [2,3], masers and parametric amplification [3], Brownian motion [4], slow neutron scattering [5], squeezed states of quantum optics [6], and electrical resistance or Ohmic friction [7].

Considerable effort has been expended in the search for methods of dealing with friction and other forms of dissipation in classical and quantum mechanics [1–11]. The mechanics developed by Newton in the 17th century can be applied to both conservative and nonconservative processes [12]. However, such classical systems cannot be quantized without first being expressed in terms of later forms of mechanics. The purpose of the present paper is to provide a general method of dealing with nonconservative forces in classical mechanics. It is hoped that the methods developed here can be extended to quantum systems as well.

Newtonian mechanics was transformed into a much more elegant and powerful formalism by Maupertuis, Euler, Lagrange, and Hamilton during the 18th and 19th centuries [13]. These more modern techniques provide a systematic approach starting with a scalar function, the Lagrangian. By using a variational principle, one can directly obtain Newtonian equations of motion, definitions of the momenta, and the Hamiltonian function. Once the Hamiltonian is known, the system becomes amenable to the techniques of quantum mechanics. Because of the importance of the variational approach, it has become the starting point for both specific calculation and general theory. As one textbook states [14], "Today most physicists would be not only willing to accept as axiomatic the existence of a variational principle but would be also loath to accept any dynamical equations that were not derivable from such a principle."

For conservative systems, variational methods are equivalent to the original mechanics used by Newton. However, while Newton's equations allow nonconservative forces, the later techniques of Lagrangian and Hamiltonian mechanics have no direct way to dealing with them. As explained by Lanczos (Ref. [15], p. 359), "Frictional forces . . . which originate from a transfer of macroscopic into microscopic motions demand an increase in the number of degrees of freedom and the application of statistical principles. They are thus automatically beyond the macroscopic variational treatment."

Over the years, a number of methods have been devised to circumvent the discrimination against nonconservative systems. One of the best known is the Rayleigh dissipation function (Ref. [16], p. 21), which can be used when frictional forces are proportional to velocity. For a particle in one dimension, Rayleigh's function is

$$\mathcal{F}=\tfrac{1}{2}\gamma\dot{x}^2 , \tag{1}$$

where $\dot{x}$ is the derivative of position. Lagrange's equation is rewritten in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}+\frac{\partial \mathcal{F}}{\partial \dot{x}}=0 . \tag{2}$$

In this case, it takes two scalar functions to specify the equation of motion. The momentum and the Hamiltonian are the same as if no friction were present, so they are of no use when attempting to quantize friction.

Another method is to use a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion [3,9]. For example, the Lagrangian

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$$L=\tfrac{1}{2}m\dot{x}^2 e^{(\gamma/m)t} \tag{3}$$

leads to the Euler-Lagrange equation

$$e^{(\gamma/m)t}(m\ddot{x}+\gamma\dot{x})=0\ , \tag{4}$$

conjugate momentum

$$p=m\dot{x}e^{(\gamma/m)t}\ , \tag{5}$$

and Hamiltonian

$$H=\frac{p^2}{2m}e^{-(\gamma/m)t}\ . \tag{6}$$

The desired equation of motion is obtained if the factor $e^{(\gamma/m)t}$ is ignored. However, the momentum and Hamiltonian do not appear to be physically meaningful.

A different technique [6,10] is to introduce an auxiliary coordinate y that describes a reverse-time system with negative friction. The Lagrangian for the combined system is

$$L=m\dot{x}\dot{y}+\tfrac{1}{2}\gamma(x\dot{y}-\dot{x}y)\ , \tag{7}$$

which leads to two equations of motion

$$m\ddot{x}+\gamma\dot{x}=0,\quad m\ddot{y}-\gamma\dot{y}=0\ , \tag{8}$$

two momenta

$$p_x=m\dot{y}-\tfrac{1}{2}\gamma y,\quad p_y=m\dot{x}+\tfrac{1}{2}\gamma x\ , \tag{9}$$

and the Hamiltonian

$$H=\frac{p_x p_y}{m}+\frac{\gamma}{2m}(yp_y-xp_x)-\frac{\gamma^2}{2m}xy\ . \tag{10}$$

The Hamiltonian leads to extraneous solutions that must be suppressed and the physical meaning of the momenta is unclear.

The most realistic approach is to include the microscopic details of the dissipation directly in the Lagrangian or Hamiltonian [1,2,4,5,7,11]. For example, if the dissipation is due to the interaction with a bath of harmonic oscillators with coordinates y_j, the following terms can be added to the Hamiltonian

$$H_{\text{bath}}=-x\sum_j c_j y_j+x^2\sum_j\frac{c_j^2}{2m_j\omega_j^2}+\sum_j\left[\frac{p_j^2}{2m_j}+\frac{1}{2}m_j\omega_j^2y_j^2\right]\ . \tag{11}$$

This method is well suited to realistic applications that can be modeled with harmonic oscillators. It is easily quantized and provides a derivation of the generalized Langevin equation. It can include the effects of driving noise and thermal equilibrium at any temperature. Because it does not deal with dissipation as a macroscopic approximation, it is well defined and easy to understand. For these reasons, it has been a valuable tool in the study of quantum dissipation. However, it is not intended to be a general method of introducing friction into classical Lagrangian mechanics. It can be complex in practice and does not allow the functional form of the frictional force to be chosen arbitrarily.

None of the above techniques exhibits the same directness and simplicity found in the mechanics of conservative systems. The method presented in the present paper will allow nonconservative forces to be calculated directly from a Lagrangian. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. The method is based on the simple observation that if the Lagrangian contains a term proportional to $(d^nx/dt^n)^2$, then the Euler-Lagrange equation will have a corresponding term proportional to $d^{2n}x/dt^{2n}$. Hence a frictional force of the form $\gamma(dx/dt)$ should follow directly from a Lagrangian containing a term proportional to the fractional derivative $(d^{1/2}x/dt^{1/2})^2$. This technique overcomes many of the objections raised for the other methods, but is not without difficulties of its own, most notably the added complexity brought on by the use of fractional derivatives.

In general, derivatives of any noninteger order are termed "fractional derivatives." Such derivatives in the Lagrangian will be seen to lead to nonconservative forces. However, mathematical techniques for dealing with derivatives of noninteger order are relatively unfamiliar. For this reason, fractional derivatives are reviewed in Sec. II. The general case of a classical Lagrangian and Hamiltonian with fractional derivatives is presented in Sec. III. Section IV applies the formalism to the example of a classical frictional force proportional to velocity. Conclusions are presented in Sec. V.

II. FRACTIONAL DERIVATIVES

This paper will make essential use of the concept of derivatives of fractional order. The history of fractional derivatives [17,18] starts in 1695, when l'Hôpital [19] suggested to Leibniz the possibility of taking a derivative of order $\frac{1}{2}$. Although the subject was also considered by Euler [20] and Laplace [21], fractional derivatives did not appear in a text until 1819, when Lacroix [22] based a definition on the usual expression for the nth derivative of a power of t:

$$\frac{d^nt^m}{dt^n}=\frac{m!}{(m-n)!}t^{m-n}\ , \tag{12}$$

where n is an integer. He defined the fractional derivative by the same formula, but for arbitrary values of n. The factorials $m!$ and $(m-n)!$ must then be interpreted as the gamma functions $\Gamma(m+1)$ and $\Gamma(m-n+1)$. This definition allows us to calculate the nth-order fractional derivative of functions that can be expressed as a power series,

$$\frac{d^nf(t)}{dt^n}=\frac{d^n}{dt^n}\sum_m a_mt^m=\sum_m a_m\frac{m!}{(m-n)!}t^{m-n}\ . \tag{13}$$

Equation (13) is not the only possible definition of fractional differentiation. Another reasonable definition is based on the expression

$$\frac{d^n e^{at}}{d(t+\infty)^n} = a^n e^{at} , \tag{14}$$

which is valid for integer-order derivatives. [The reason for writing the derivative as $d^n/d(t+\infty)^n$ will soon become apparent.] If we consider noninteger values of n, the formula can be chosen to define the fractional derivative of an exponential. It also defines the fractional derivative of any function that can be expressed as a sum of exponentials,

$$\frac{d^n}{d(t+\infty)^n} f(t) = \frac{d^n}{d(t+\infty)^n} \sum_m c_m e^{a_m t} = \sum_m c_m a_m^n e^{a_m t} . \tag{15}$$

This definition of fractional derivatives was proposed by Liouville [23] in 1832 and was rediscovered by Ramanujan [24] in 1914. Fourier [25] suggested a similar definition using Fourier transforms in 1822.

The study of fractional derivatives would no doubt be more popular today were it not for the fact that the definition in terms of exponentials, Eq. (15), is not equivalent to the definition in terms of powers, Eq. (13). There is no equally simple definition that applies both to functions expressed as exponentials and to functions expressed as powers. In order to obtain a definition that is as general as possible, it has become conventional to use an integral representation discovered by Liouville [23] and extended by Riemann [26]. Define the fractional integral of order v by

$$\frac{d^{-v} f(t)}{d(t-c)^{-v}} = \frac{1}{\Gamma(v)} \int_c^t (t-t')^{v-1} f(t') dt' \quad [\mathrm{Re}(v) > 0] . \tag{16}$$

If n is the smallest integer greater than $\mathrm{Re}(u)$ and $v = n - u$, then the fractional derivative of order u is defined by

$$\frac{d^u f(t)}{d(t-c)^u} = \frac{d^n}{dt^n} \frac{d^{-v} f(t)}{d(t-c)^{-v}} . \tag{17}$$

The above notation, which will be used throughout the paper, follows Oldham and Spanier [17]. Another common notation was introduced by Davis [27] and is used by Miller and Ross [18]:

$${}_c D_t^u f(t) = \frac{d^u f(t)}{d(t-c)^u} . \tag{18}$$

Both these notations emphasize that the fractional derivative of a function is not determined by the behavior of the function at the single value t, but depends on the values of the function over the entire interval c to t, just as a definite integral depends on values throughout the interval of integration. It can be shown [17,18] that for $c = -\infty$, the definition becomes consistent with Liouville's definition in terms of exponentials. For $c = 0$, the definition agrees with Lacroix's definition in terms of powers. If u is an integer, then $d^u x / d(t-c)^u$ is simply the usual derivative $d^u x / dt^u$ and the constant c can be omitted.

The remainder of this section is a summary of some results of fractional calculus for later use. More details can be found in Refs. [17,18]. In applications, integer-order derivatives with respect to t may be denoted with dots, so that $\dot{x} = dx/dt$ and $\ddot{x} = d^2 x / dt^2$. Derivatives of arbitrary order with respect to t will sometimes be indicated by a subscript or superscript in parentheses: $x_{(u,a)} = x^{(u,a)} = d^u x / d(t-a)^u$. If the constant is omitted, it is assumed to be zero: $x_{(1/2)} = d^{1/2} x / dt^{1/2}$.

Fractional derivatives are linear operators obeying

$$\frac{d^u}{d(t-c)^u}[A f(t) + B g(t)] = A \frac{d^u}{d(t-c)^u} f(t) + B \frac{d^u}{d(t-c)^u} g(t) . \tag{19}$$

The fractional derivative of a constant A is not zero in general, but is

$$\frac{d^u A}{d(t-c)^u} = \frac{A(t-c)^{-u}}{\Gamma(1-u)} . \tag{20}$$

However, the fractional derivative of zero is always zero:

$$\frac{d^u 0}{d(t-c)^u} = 0 . \tag{21}$$

Fractional derivatives can be approximated by finite differences. A number of choices can be found in the literature [17,28,29], since different finite-difference expressions can converge to the same limit. For the derivative in terms of backward differences we will use

$$\frac{d^u x}{d(t-a)^u} = \lim_{N \to \infty} \left[\frac{t-a}{N} \right]^{-u} \times \sum_{m=0}^{N} (-1)^m \begin{Bmatrix} u \\ m \end{Bmatrix} x \left[t - m \frac{t-a}{N} \right] , \tag{22}$$

where

$$\begin{Bmatrix} u \\ m \end{Bmatrix} = \frac{u!}{(u-m)! m!} = \frac{\Gamma(u+1)}{\Gamma(u-m+1)\Gamma(m+1)} . \tag{23}$$

This formula is an extension of the usual backward difference for integer-order derivatives,

$$\frac{d^n x}{dt^n} = \lim_{\Delta t \to 0} (\Delta t)^{-n} \sum_{m=0}^{n} (-1)^m \begin{Bmatrix} n \\ m \end{Bmatrix} x(t - m \Delta t) . \tag{24}$$

For large N, it agrees with the definition used by Oldham and Spanier [17], but provides a better approximation when N is small. It can be shown [17] that the finite-difference formula is equivalent to the definition in terms of an integral using Eqs. (16) and (17). For forward differences we will use

$$\frac{d^u x}{d(t-b)^u} = \lim_{N \to \infty} \left[\frac{b-t}{N} \right]^{-u} \times \sum_{m=0} (-1)^{m+u} \begin{Bmatrix} u \\ m \end{Bmatrix} x \left[t + m \frac{b-t}{N} \right] . \tag{25}$$

The theory of fractional integration is a fully developed field, although all we will need are some relationships for antiderivatives. The antiderivative of order u is written as a derivative of order $-u$ and satisfies

$$\frac{d^u}{d(t-c)^u}\frac{d^{-u}f(t)}{d(t-c)^{-u}}=f(t)\ . \qquad (26)$$

Derivatives and antiderivatives obey the composition rule

$$\frac{d^u}{d(t-c)^u}\left[\frac{d^v}{d(t-c)^v}f(t)\right]=\frac{d^{u+v}}{d(t-c)^{u+v}}f(t) \qquad (27)$$

when $v\leq 0$ or $uv\geq 0$.

A formula we will need later is integration by parts of a fractional derivative. The conventional formula for integer-order derivatives is

$$\int_a^b\frac{d^nf(t)}{dt^n}g(t)dt-(-1)^n\int_a^b f(t)\frac{d^ng(t)}{dt^n}dt=\sum_{k=0}^{n-1}(-1)^k\frac{d^{n-k-1}f(t)}{dt^{n-k-1}}\frac{d^kg(t)}{dt^k}\Bigg|_a^b\ . \qquad (28)$$

When either $d^kf/dt^k=0$ or $d^kg/dt^k=0$ for $k=0$ to $n-1$, the formula becomes

$$\int_a^b\frac{d^nf(t)}{dt^n}g(t)dt=(-1)^n\int_a^b f(t)\frac{d^ng(t)}{dt^n}dt\ . \qquad (29)$$

Love and Young [30] have obtained a fractional-order formula using the functions $f_v^+(a,x)$ and $f_v^-(x,b)$, which can be written as fractional antiderivatives

$$\begin{aligned}f_v^+(a,x)&=\frac{1}{\Gamma(u)}\int_a^x f(t)(x-t)^{v-1}dt=\frac{d^{-v}f(x)}{d(x-a)^{-v}}\ ,\\ f_v^-(x,b)&=\frac{1}{\Gamma(u)}\int_x^b f(t)(t-x)^{v-1}dt\\ &=(-1)^v\frac{1}{\Gamma(u)}\int_b^x f(t)(x-t)^{v-1}dt\\ &=(-1)^v\frac{d^{-v}f(x)}{d(x-b)^{-v}}\ .\end{aligned} \qquad (30)$$

They prove

$$\int_a^b f_v^+(a,x)g(x)dx=\int_a^b f(x)g_v^-(x,b)dx \qquad (31)$$

for $0<v<1$. In our notation, this equation becomes

$$\int_a^b\frac{d^{-v}f(t)}{d(t-a)^{-v}}g(t)dt=(-1)^v\int_a^b f(t)\frac{d^{-v}g(t)}{d(t-b)^{-v}}dt\ . \qquad (32)$$

To obtain a general formula for integration by parts for order u, we choose n to be the smallest integer greater than u and let $v=n-u$. Then application of Eq. (32) followed by Eq. (29) yields the desired result

$$\int_a^b\frac{d^uf(t)}{d(t-a)^u}g(t)dt=(-1)^{-u}\int_a^b f(t)\frac{d^ug(t)}{d(t-b)^u}dt \qquad (33)$$

provided that $d^kf/dt^k=0$ or $d^kg/dt^k=0$ for $k=0$ to $n-1$.

III. GENERALIZED MECHANICS WITH FRACTIONAL DERIVATIVES

A. Introduction and notation

In Newtonian mechanics, equations of motion are usually expressed using derivatives of second order or lower. The corresponding Lagrangians have derivatives no higher than first order. Ostrogradsky [31] was the first to publish a generalization of Lagrangian and Hamiltonian mechanics to include arbitrarily high-order derivatives. Dynamical equations with higher-order derivatives can be used to describe particles with internal structure, such as spin or internal motion [32]. The formalism was extended to quantum electrodynamics by Bopp [33] and Podolsky [34] and to quantum field theory by Green [35]. Generalized mechanics is reviewed in Ref. [36] and recent applications are described in Refs. [37,38]. The present work can be considered to be a further generalization of mechanics to include fractional derivatives of all orders.

The starting point for generalized mechanics is a Lagrangian, which is a function of coordinates x_r, the parameter t, and derivatives of x_r with respect to t. The subscript $r=1,\ldots,R$ indicates the particular coordinate (for example, $x_1=x$, $x_2=y$, $x_3=z$). We will not restrict the derivatives to integer order, but will allow any fractional or higher-order derivative. If the Lagrangian is a function of the coordinate x_r and N different derivatives of x_r, then we will use $s(n)$ to indicate the order of the nth derivative, where $n=1,\ldots,N$. For example, if the lowest-order derivative is $d^{1/2}x/d(t-b)^{1/2}$, then $s(1)=\frac{1}{2}$. For each order of derivative in the Lagrangian, we define generalized coordinates by

$$q_{r,s(n)}=q_{r,s(n),b}=\frac{d^{s(n)}x_r}{d(t-b)^{s(n)}}\ , \qquad (34)$$

where $s(n)$ can be any non-negative real number (or complex number with $\mathrm{Re}[s(n)]\geq 0$). We define $s(0)$ to be 0, so that $q_{r,s(0)}$ denotes the coordinate x_r.

If a variational principle is applied over the interval $t=a$ to b, we will find that the Lagrangian can also be a function of a second type of coordinate

$$q_{r,s'(n),a}=\frac{d^{s'(n)}x_r}{d(t-a)^{s'(n)}}\ . \qquad (35)$$

To avoid overburdening an already tedious notation, we will derive all results assuming a Lagrangian that does not contain any derivatives of this second type. The straightforward extension of each final result to Lagrangians with both types of derivatives will then be provided. In derivations that use only coordinates defined by Eq. (34), the subscript b on the coordinates will be omitted.

The notation $L(\{q_{r,s(n)}\},t)$ will be used to indicate that the Lagrangian is a function of the parameter t and the set of all $q_{r,s(n)}$ for $r=1,\ldots,R$ and $n=0,\ldots,N$. The notation $L(\{q_{r,s(n),a},q_{r,s'(n),b}\},t)$ will be used for a Lagrangian that is a function of both types of coordinates. Because summations over r will always be over all values, we will use the usual convention of summing over repeated indices. However, we will not be able to use the summation convention for n in all cases, so all summations over n will be indicated explicitly.

B. Fractional calculus of variations using finite differences

The most intuitive derivation of the Euler-Lagrange equation is an extension of the original method used by Euler. The details of the original method can be found in Chap. 2 of [15]. Euler's method has been criticized for not being entirely rigorous, since it exchanges the order of sums and limits. However, the method is useful for introducing a variational principle and it does not rely on formal results from the calculus of variations. It also provides a direct method for numerical calculation. A more rigorous derivation is provided in Sec. III C.

Our goal is to derive a variational principle based on the generalized Lagrangian $L(\{q_{r,s(n),a},q_{r,s'(n),b}\},t)$. We will start by finding the extremum of the integral of a Lagrangian with a representative fractional-derivative term

$$J=\int_a^b L\left[\frac{d^u x}{d(t-a)^u},t\right]dt \; . \tag{36}$$

The result will then be extended to the general Lagrangian. We must find a function $x(t)$ that will make the integral an extremum. We vary $x(t)$ inside the interval (a,b), keeping it fixed at the end points $t=a$ and b.

We follow Euler's original method, which makes a finite-difference approximation J_s to the integral and derivatives. First divide the interval into n equal increments Δt at the following points: $t_0=a,t_1,t_2,\ldots,t_n=b$. The fixed increment Δt is $\Delta t=(t_n-t_0)/n$. For the backward difference at t_j, the same interval can be expressed as $\Delta t=(t_j-a)/j$ and for the forward difference at t_j it can be written $\Delta t=(b-t_j)/(n-j)$. In terms of these differences, the backward fractional difference is

$$\left[\frac{d^u x}{d(t-a)^u}\right]_{t=t_j}\approx\left[\frac{\Delta^u x}{\Delta(t-a)^u}\right]_{t=t_j}=x_j^{u,a}=\frac{1}{(\Delta t)^u}\sum_{m=0}^{j}(-1)^m\begin{Bmatrix}u\\m\end{Bmatrix}x_{j-m} \tag{37}$$

and the forward difference is

$$\left[\frac{d^u x}{d(t-b)^u}\right]_{t=t_j}\approx\left[\frac{\Delta^u x}{\Delta(t-b)^u}\right]_{t=t_j}=x_j^{u,b}=\frac{1}{(\Delta t)^u}\sum_{m=0}^{n-j}(-1)^{m+u}\begin{Bmatrix}u\\m\end{Bmatrix}x_{j+m} \; . \tag{38}$$

To apply Euler's method to the integral, replace the derivative by the forward finite difference $x_j^{u,b}$ and then express the integral as the sum

$$J_s=\sum_{j=0}^{n}L(x_j^{u,b},t_j)\Delta t \; . \tag{39}$$

Following Euler's method [15], we perform the variation by setting

$$0=\frac{\partial J_s}{\partial x_k} \; . \tag{40}$$

If we carry out the differentiation, we find

$$0=\sum_{j=0}^{n}\left[\left[\frac{\partial L}{\partial x^{(u,b)}}\right]_{t=t_j}\frac{\partial x_j^{u,b}}{\partial x_k}+\left[\frac{\partial L}{\partial t}\right]_{t=t_j}\frac{\partial t_j}{\partial x_k}\right] \; . \tag{41}$$

Since the second term is zero, we have

$$\begin{aligned}
0&=\sum_{j=0}^{n}\left[\frac{\partial L}{\partial x^{(u,b)}}\right]_{t=t_j}\frac{\partial}{\partial x_k}\sum_{m=0}^{n-j}\frac{(-1)^{m+u}}{(\Delta t)^u}\begin{Bmatrix}u\\m\end{Bmatrix}x_{j+m}\\
&=\sum_{j=0}^{n}\frac{1}{(\Delta t)^u}\sum_{m=0}^{n-j}(-1)^{m+u}\begin{Bmatrix}u\\m\end{Bmatrix}\left[\frac{\partial L}{\partial x^{(u,b)}}\right]_{t=t_j}\delta_{j+m,k}\\
&=(-1)^u\frac{1}{(\Delta t)^u}\sum_{m=0}^{k}(-1)^m\begin{Bmatrix}u\\m\end{Bmatrix}\left[\frac{\partial L}{\partial x^{(u,b)}}\right]_{t=t_{k-m}}\\
&=\left[(-1)^u\frac{\Delta^u}{\Delta(t-a)^u}\frac{\partial L}{\partial x^{(u,b)}}\right]_{t=t_k} \; .
\end{aligned} \tag{42}$$

In the limit $\Delta t\to 0$, the finite difference becomes a derivative. Since the expression is true for any k in the integration interval and t_k comes arbitrarily close to any value of t in the interval, the following equation must hold:

$$(-1)^u\frac{d^u}{d(t-a)^u}\frac{\partial L}{\partial x^{(u,b)}}=0 \; . \tag{43}$$

If L is a function of $x^{(u,a)}$ instead of $x^{(u,b)}$, then a similar derivation shows that

$$(-1)^{-u}\frac{d^u}{d(t-b)^u}\frac{\partial L}{\partial x^{(u,a)}}=0 \; . \tag{44}$$

If the derivation is repeated for the general Lagrangian $L(\{q_{r,s(n),a},q_{r,s'(n),b}\},t)$, using the notation of Sec. III A, we find the generalization of the Euler-Lagrange equation

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{d^{s(n)}}{d(x-a)^{s(n)}}\frac{\partial L}{\partial q_{r,s(n),b}}+\sum_{n=1}^{N'}(-1)^{-s'(n)}\frac{d^{s'(n)}}{d(x-b)^{s'(n)}}\frac{\partial L}{\partial q_{r,s'(n),a}}=0 \; . \tag{45}$$

C. Euler-Lagrange equation

Derivations of the type presented in Sec. III B are open to the criticism that the interchange of limits and summations may not be justified (Ref. [15], p. 54). The following derivation of the Euler-Lagrange equation follows the same pattern as in the conventional calculus of variations used in classical mechanics (Ref. [16], Chap. 2). Start with the integral

$$J=\int_a^b L(\{q_{r,s(n)}\},t)dt \ . \tag{46}$$

Following the procedure described in Sec. III A, we will initially assume that L contains only coordinates as defined in Eq. (34). Consider the coordinates $q_{r,s(n)}$ to be functions of both the variable t and a parameter α, which is varied over all paths from $t=a$ to b to make the integral an extremum. The variation is zero at $t=a$ and b. J is then a function α,

$$J(\alpha)=\int_a^b L(\{q_{r,s(n)}(t,\alpha)\},t)dt \ . \tag{47}$$

This function will be an extremum if

$$\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0}=0 \ . \tag{48}$$

Performing the variation on the integral results in

$$\frac{\partial J(\alpha)}{\partial \alpha}d\alpha=\int_a^b \sum_{n=0}^{N}\frac{\partial L}{\partial q_{r,s(n)}}\frac{\partial q_{r,s(n)}}{\partial \alpha}d\alpha\, dt \ , \tag{49}$$

where summation over the index r is implied by our summation convention. We next rewrite this expression so that each term in the integrand contains a factor of $\partial q_{r,s(0)}/\partial\alpha$, which vanishes for $t=a$ and b. The $n=0$ term is already in this form. The other terms can be converted by using integration by parts. For each term containing a derivative, we can apply the formula for integration by parts, Eq. (33), to get

$$\begin{aligned}\delta J&=\int_a^b \sum_{n=0}^{N}\frac{\partial L}{\partial q_{r,s(n)}}\frac{\partial q_{r,s(n)}}{\partial \alpha}d\alpha\, dt\\ &=\int_a^b \sum_{n=0}^{N}\frac{\partial L}{\partial q_{r,s(n)}}\frac{\partial^{s(n)}}{\partial (t-b)^{s(n)}}\frac{\partial q_{r,s(0)}}{\partial \alpha}d\alpha\, dt\\ &=\int_a^b \sum_{n=0}^{N}\left[(-1)^{s(n)}\frac{d^{s(n)}}{d(t-a)^{s(n)}}\right.\\ &\qquad\left.\times\frac{\partial L}{\partial q_{r,s(n)}}\right]\delta q_{r,s(0)}dt \ ,\end{aligned} \tag{50}$$

where

$$\delta q_{r,s(0)}=\frac{\partial q_{r,s(0)}}{\partial \alpha}d\alpha, \quad \delta J=\frac{\partial J(\alpha)}{\partial \alpha}d\alpha \ . \tag{51}$$

Since the variations are independent, δJ can only be zero if the coefficients of $\delta q_{r,s(0)}$ are zero. Hence we obtain the generalized Euler-Lagrange equation

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{d^{s(n)}}{d(x-a)^{s(n)}}\frac{\partial L}{\partial q_{r,s(n)}}=0 \ . \tag{52}$$

If the Lagrangian is a function of coordinates defined by both Eqs. (34) and (35), a similar derivation provides

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{d^{s(n)}}{d(x-a)^{s(n)}}\frac{\partial L}{\partial q_{r,s(n),b}} +\sum_{n=1}^{N'}(-1)^{-s'(n)}\frac{d^{s'(n)}}{d(x-b)^{s'(n)}}\frac{\partial L}{\partial q_{r,s'(n),a}}=0 \ , \tag{53}$$

in agreement with Eq. (45).

D. Hamilton's equations

The derivation of Hamilton's equations begins with the introduction of the generalized momenta. As before, we first consider only coordinates defined by Eq. (34) and then generalize the final results. Using a bit of foresight, define

$$\begin{aligned}p_{r,s(n)}&=p_{r,s(n),b}\\ &=\sum_{k=0}^{N-n-1}(-1)^{s(k+n+1)-s(n+1)}\\ &\qquad\times\frac{d^{s(k+n+1)-s(n+1)}}{d(t-a)^{s(k+n+1)-s(n+1)}}\\ &\qquad\times\left[\frac{\partial L}{\partial q_{r,s(k+n+1)}}\right] ,\end{aligned} \tag{54}$$

where $n=0,\ldots,N-1$. The Hamiltonian is defined to be

$$H=\sum_{n=1}^{N}q_{r,s(n)}p_{r,s(n-1)}-L \ , \tag{55}$$

where the summation convention again implies summation over r. We next must show that these definitions satisfy a set of generalized Hamilton equations that are equivalent to the Euler-Lagrange equation. The proof follows the derivation used by Goldstein (Ref. [16], Chap. 7). First differentiate the Hamiltonian

$$dH=\sum_{n=1}^{N}q_{r,s(n)}dp_{r,s(n-1)}+\sum_{n=1}^{N}p_{r,s(n-1)}dq_{r,s(n)} -\sum_{n=0}^{N}\frac{\partial L}{\partial q_{r,s(n)}}dq_{r,s(n)}-\frac{\partial L}{\partial t}dt \ . \tag{56}$$

Now split off the $n=0$ term from the third sum in the differential of the Hamiltonian and substitute the definition of the momenta

$$\begin{aligned}dH=&\sum_{n=1}^{N}q_{r,s(n)}dp_{r,s(n-1)}+\sum_{n=1}^{N}\left[\sum_{k=0}^{N-n}(-1)^{s(k+n)-s(n)}\frac{d^{s(k+n)-s(n)}}{d(t-a)^{s(k+n)-s(n)}}\left[\frac{\partial L}{\partial q_{r,s(k+n)}}\right]\right]dq_{r,s(n)}\\ &-\frac{\partial L}{\partial q_{r,s(0)}}dq_{r,s(0)}-\sum_{n=1}^{N}\frac{\partial L}{\partial q_{r,s(n)}}dq_{r,s(n)}-\frac{\partial L}{\partial t}dt \ .\end{aligned} \tag{57}$$

The $k=0$ term in the second summation cancels the last summation. In the second sum, the $N=n$ term is then zero, so we can write

$$dH=\sum_{n=1}^{N} q_{r,s(n)}dp_{r,s(n-1)}+\sum_{n=1}^{N-1}\left[\sum_{k=1}^{N-n}(-1)^{s(k+n)-s(n)}\frac{d^{s(k+n)-s(n)}}{d(t-a)^{s(k+n)-s(n)}}\left[\frac{\partial L}{\partial q_{r,s(k+n)}}\right]\right]dq_{r,s(n)}$$
$$-\frac{\partial L}{\partial q_{r,s(0)}}dq_{r,s(0)}-\frac{\partial L}{\partial t}dt\ . \tag{58}$$

The Euler-Lagrange equation, Eq. (52), can be rewritten as

$$\frac{\partial L}{\partial q_{r,s(0)}}=-\sum_{k=1}^{N}(-1)^{s(k)}\frac{d^{s(k)}}{d(t-a)^{s(k)}}\left[\frac{\partial L}{\partial q_{r,s(k)}}\right] \tag{59}$$

and then substituted for the third term to get

$$dH=\sum_{n=1}^{N} q_{r,s(n)}dp_{r,s(n-1)}+\sum_{n=1}^{N-1}\left[\sum_{k=1}^{N-n}(-1)^{s(k+n)-s(n)}\frac{d^{s(k+n)-s(n)}}{d(t-a)^{s(k+n)-s(n)}}\left[\frac{\partial L}{\partial q_{r,s(k+n)}}\right]\right]dq_{r,s(n)}$$
$$+\left[\sum_{k=1}^{N}(-1)^{s(k)}\frac{d^{s(k)}}{d(t-a)^{s(k)}}\left[\frac{\partial L}{\partial q_{r,s(k)}}\right]\right]dq_{r,s(0)}-\frac{\partial L}{\partial t}dt\ . \tag{60}$$

The second and third summations can be combined to give

$$dH=\sum_{n=1}^{N} q_{r,s(n)}dp_{r,s(n-1)}+\sum_{n=0}^{N-1}\left[\sum_{k=1}^{N-n}(-1)^{s(k+n)-s(n)}\frac{d^{s(k+n)-s(n)}}{d(t-a)^{s(k+n)-s(n)}}\left[\frac{\partial L}{\partial q_{r,s(k+n)}}\right]\right]dq_{r,s(n)}-\frac{\partial L}{\partial t}dt\ , \tag{61}$$

which may be rewritten as

$$dH=\sum_{n=1}^{N} q_{r,s(n)}dp_{r,s(n-1)}-\frac{\partial L}{\partial t}dt$$
$$+\sum_{n=0}^{N-1}(-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}}{d(t-a)^{s(n+1)-s(n)}}$$
$$\times\left[\sum_{k=0}^{N-n-1}(-1)^{s(k+n+1)-s(n+1)}\frac{d^{s(k+n+1)-s(n+1)}}{d(t-a)^{s(k+n+1)-s(n+1)}}\left[\frac{\partial L}{\partial q_{r,s(k+n+1)}}\right]\right]dq_{r,s(n)}\ . \tag{62}$$

After substituting for momentum we obtain

$$dH=\sum_{n=0}^{N-1} q_{r,s(n+1)}dp_{r,s(n)}$$
$$+\sum_{n=0}^{N-1}(-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}}{d(t-a)^{s(n+1)-s(n)}}$$
$$\times p_{r,s(n)}dq_{r,s(n)}-\frac{\partial L}{\partial t}dt\ . \tag{63}$$

A term-by-term comparison to

$$dH=\sum_{n=0}^{N-1}\frac{\partial H}{\partial q_{r,s(n)}}dq_{r,s(n)}$$
$$+\sum_{n=0}^{N-1}\frac{\partial H}{\partial p_{r,s(n)}}dp_{r,s(n)}+\frac{\partial H}{\partial t}dt \tag{64}$$

yields the generalized Hamilton equations

$$\frac{\partial H}{\partial q_{r,s(n)}}=(-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}}{d(t-a)^{s(n+1)-s(n)}}p_{r,s(n)}\ ,$$
$$\frac{\partial H}{\partial p_{r,s(n)}}=q_{r,s(n+1)}\ , \tag{65}$$
$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}\ .$$

If the Lagrangian is a function of coordinates defined by both Eqs. (34) and (35), then we must define additional momenta

$$p_{r,s'(n),a}=\sum_{k=0}^{N'-n-1}(-1)^{-[s'(k+n+1)-s'(n+1)]}$$
$$\times\frac{d^{s'(k+n+1)-s'(n+1)}}{d(t-b)^{s'(k+n+1)-s'(n+1)}}$$
$$\times\left[\frac{\partial L}{\partial q_{r,s'(k+n+1),a}}\right]\ , \tag{66}$$

where $n=0,\ldots,N'-1$. The Hamiltonian is defined to be

$$H=\sum_{n=1}^{N} q_{r,s(n),b}p_{r,s(n-1),b} + \sum_{n=1}^{N'} q_{r,s'(n),a}p_{r,s'(n-1),a} - L \ . \tag{67}$$

A derivation similar to that for Eq. (65) provides the additional equations

$$\frac{\partial H}{\partial q_{r,s'(n),a}}=(-1)^{-[s'(n+1)-s'(n)]} \times \frac{d^{s'(n+1)-s'(n)}}{d(t-b)^{s'(n+1)-s'(n)}}p_{r,s'(n),a} \ ,$$
$$\frac{\partial H}{\partial p_{r,s'(n),a}}=q_{r,s'(n+1),a} \ . \tag{68}$$

E. Time dependence and nonconservative systems

The time dependence of the Hamiltonian can be determined by writing the total derivative of H as

$$\frac{dH}{dt}=\sum_{n=0}^{N-1}\frac{\partial H}{\partial q_{r,s(n)}}\frac{dq_{r,s(n)}}{dt}+\sum_{n=0}^{N-1}\frac{\partial H}{\partial p_{r,s(n)}}\frac{dp_{r,s(n)}}{dt} + \sum_{n=0}^{N'-1}\frac{\partial H}{\partial q_{r,s'(n),a}}\frac{dq_{r,s'(n),a}}{dt} + \sum_{n=0}^{N'-1}\frac{\partial H}{\partial p_{r,s'(n),a}}\frac{dp_{r,s'(n),a}}{dt}+\frac{\partial H}{\partial t} \ . \tag{69}$$

If all derivatives are of integer order, then generalized coordinates can be chosen so that $s(n+1)-s(n)=1$. In this case, Hamilton's equations can be used to obtain

$$\frac{dH}{dt}=-\sum_{n=0}^{N-1}\frac{dp_{r,s(n)}}{dt}\frac{dq_{r,s(n)}}{dt}+\sum_{n=0}^{N-1}\frac{dq_{r,s(n)}}{dt}\frac{dp_{r,s(n)}}{dt} - \sum_{n=0}^{N'-1}\frac{dp_{r,s'(n),a}}{dt}\frac{dq_{r,s'(n),a}}{dt} + \sum_{n=0}^{N'-1}\frac{dq_{r,s'(n),a}}{dt}\frac{dp_{r,s'(n),a}}{dt}+\frac{\partial H}{\partial t} \ , \tag{70}$$

so that

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t} \ . \tag{71}$$

Hence, if the Lagrangian is not an explicit function of time and all derivatives are of integer order, then the Hamiltonian is a constant of the motion. However, if there are fractional derivatives in the Lagrangian, then we will not have $s(n+1)-s(n)=1$ for all n and instead we find

$$\frac{dH}{dt}=\sum_{n=0}^{N-1}(-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}p_{r,s(n)}}{d(t-a)^{s(n+1)-s(n)}}\frac{dq_{r,s(n)}}{dt}+\sum_{n=0}^{N-1}\frac{d^{s(n+1)-s(n)}q_{r,s(n)}}{d(t-b)^{s(n+1)-s(n)}}\frac{dp_{r,s(n)}}{dt} + \sum_{n=0}^{N'-1}(-1)^{-[s'(n+1)-s'(n)]}\frac{d^{s'(n+1)-s'(n)}p_{r,s'(n),a}}{d(t-b)^{s'(n+1)-s'(n)}}\frac{dq_{r,s'(n),a}}{d(t-b)}+\sum_{n=0}^{N'-1}\frac{d^{s'(n+1)-s'(n)}q_{r,s'(n),a}}{d(t-a)^{s'(n+1)-s'(n)}}\frac{dp_{r,s'(n),a}}{dt}+\frac{\partial H}{\partial t} \ . \tag{72}$$

In this case, the terms do not cancel except for special cases. Therefore, a Hamiltonian with fractional derivatives usually is not a constant of the motion and the system is nonconservative.

IV. APPLICATION TO LINEAR FRICTION

The formalism of the preceding section can be illustrated with an example. For simplicity, we will choose a Lagrangian that is a function of coordinates defined by Eq. (34). We will consider the limiting case in which $a \to b$ while keeping $a<b$. Hence, all fractional derivatives we encounter can be approximated by derivatives of the form $d^u/d(t-b)^u$.

The three terms in the Lagrangian

$$L=\tfrac{1}{2}m\left[\frac{dx}{dt}\right]^2-V(x)+i\tfrac{1}{2}\gamma\left[\frac{d^{1/2}x}{d(t-b)^{1/2}}\right]^2 = \tfrac{1}{2}m\dot{x}^2-V(x)+i\tfrac{1}{2}\gamma x_{(1/2,b)}^2 \tag{73}$$

represent kinetic energy, potential energy, and linear friction energy. We can apply the methods of Sec. III by choosing $N=2$, $s(0)=0$, $s(1)=\frac{1}{2}$, and $s(2)=1$. The Lagrangian can be written as a function of the generalized coordinates

$$L=\tfrac{1}{2}mq_1^2-V(q_0)+i\tfrac{1}{2}\gamma q_{1/2}^2 \ . \tag{74}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial q_0}+i\frac{d^{1/2}}{d(t-b)^{1/2}}\frac{\partial L}{\partial q_{1/2}}-\frac{d}{dt}\frac{\partial L}{\partial q_1}=0 \ , \tag{75}$$

which becomes

$$m\ddot{x} = -\gamma\dot{x} - \frac{\partial V}{\partial x} , \tag{76}$$

since

$$\frac{d^{1/2}}{d(t-b)^{1/2}}\frac{d^{1/2}}{d(t-b)^{1/2}}x = \frac{d}{d(t-b)}x = \frac{d}{dt}x = \dot{x} . \tag{77}$$

The momenta are

$$\begin{aligned} p_0 &= \left[\frac{\partial L}{\partial q_{1/2}}\right] + i\frac{d^{1/2}}{d(t-b)^{1/2}}\left[\frac{\partial L}{\partial q_1}\right] \\ &= i\gamma x_{(1/2,b)} + imx_{(3/2,b)} , \\ p_{1/2} &= \left[\frac{\partial L}{\partial q_1}\right] = m\dot{x} . \end{aligned} \tag{78}$$

The Hamiltonian is

$$\begin{aligned} H &= q_{1/2}p_0 + q_1 p_{1/2} - L \\ &= \frac{p_{1/2}^2}{2m} + q_{1/2}p_0 + V - i\tfrac{1}{2}\gamma q_{1/2}^2 \end{aligned} \tag{79}$$

and Hamilton's equations are

$$\begin{aligned} \frac{\partial H}{\partial q_0} &= i\frac{d^{1/2}}{d(t-b)^{1/2}}p_0, \quad \frac{\partial H}{\partial p_0} = q_{1/2} , \\ \frac{\partial H}{\partial q_{1/2}} &= i\frac{d^{1/2}}{d(t-b)^{1/2}}p_{1/2}, \quad \frac{\partial H}{\partial p_{1/2}} = q_1 . \end{aligned} \tag{80}$$

The first of Hamilton's equations yields the Euler-Lagrange equation, the second is an identity, and the remaining two equations are equivalent to the definition of the momenta. This simple example illustrates the technique, but does not attempt to deal with the complications of more realistic scenarios that might include driving noise, equilibrium at finite temperatures, or more general frictional forces.

V. CONCLUSION

The example in Sec. IV shows that it is possible to construct a Lagrangian that describes a classical frictional force proportional to velocity. The Eulger-Lagrange equation is the familiar equation of motion from Newtonian mechanics. By using fractional derivatives of various orders, it possible to choose Lagrangians that result in a wide range of dissipative Euler-Lagrange equations. These Lagrangians will typically describe nonconservative forces involving fractional derivatives, rather than the functions more commonly used to describe dissipation. Hence we are presented with possibilities for dissipative equations, but also an increase in the complexity of the mathematics needed to deal with them. Because the method leads directly to a Hamiltonian, it may find application in dissipative quantum processes, although the quantization of fractional-derivative Hamiltonian systems may be correspondingly complex.

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